ISRAEL JOURNAL OF MATHEMATICS, Vol. 56, No. 2, 1986

ON REDUCED CONVEX BODIES

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ABSTRACT

A convex body $K \subset R^d$ is called reduced if for each convex body $K' \subset K$, $K' \neq K$, the width of K' is less than the width of K. We prove that reduced body K is of constant width if (i) the body K has a supporting sphere almost everywhere in ∂K. (The radius of the sphere may vary with the point in ∂K; the condition (i) and strict convexity do not imply each other.)

§1. The results

1.1. In 1978, Heil posed the following problem [4, Problem 27].

A convex body $K \subset R^d$ is called reduced if for each convex body $K' \subset K$, $K' \neq K$, the width of K' is less than the width of K. Is it true that each strictly convex reduced body K is of constant width? (Strict convexity means that ∂K contains no segments. Obviously each body of constant width is reduced.)

For dimension $d = 2$, the Heil problem was solved (in the affirmative) in [2]. Some other related facts are as follows. If a convex (not necessarily strictly convex) reduced body has a smooth boundary then it is of constant width (and for that is strictly convex in fact). That was indicated by Heil [5, §2] in 1978 and proved by Groemer [3, §5] in 1983. An example of the equilateral triangle shows that there exist convex reduced bodies not of constant width.

Here we deal with the problem in which the strict convexity is substituted by an almost spherical convexity defined below.

1.2. Let $C \subset R^d$ be a compact convex d-dimensional body. A point $p \in \partial C$ will be called *regular* if C has unique supporting hyperplane at p.

† Supported by an N.S.E.R.C. Grant of Canada.
Received September 15, 1984

Let $p \in \partial C$ and let H be a supporting hyperplane of C at p. Suppose there exists a compact ball B of radius R tangent to H at p and such that $C \cap N \subset B$ for a (d-dimensional) neighbourhood N of p. We will say then that C has an *R-support* (∂B) at p or that the sphere ∂B supports C at p. If C is supported by a sphere almost everywhere in ∂C (the radius of the sphere can vary from point to point), we will call C *almost spherically convex*. (Note that if C is supported by a sphere at each point of ∂C then C is clearly strictly convex.) We prove the following theorem.

1.3. THEOREM. *Let $A \subset R^d$ be a compact d-dimensional reduced almost spherically convex body. Then A is of constant width.*

The theorem follows easily (see 1.8) from Theorems 1.6 and 1.7 below.

It is known that a body of a constant width W has a spherical support (of radius W) at each of its boundary points. Therefore Theorem 1.3 is a characterization of constant width.

Note that strict convexity and almost spherical convexity do not follow from each other. An example of an almost spherically convex but not strictly convex body is as follows.

Let x, y, z be Cartesian coordinates in R^3. Define a surface S by the equations

$$x = (1 - y^2)\sin 2\psi, \qquad z = (1 - y^2)(1 + \cos 2\psi)$$

where the parameters ψ, y vary in the open rectangle $-\pi/2 < \psi < \pi/2$, $-1 < y < 1$. Thus this surface is formed by a parabola passing through the points $(0, -1, 0)$ and $(0, 1, 0)$ as its vertex slides along the circumference of unit radius in the xz-plane centered at the point $(0, 0, 1)$.

A straightforward calculation shows that the Gauss curvature of S is positive. Therefore the body bounded by S and by the segment with ends $(0, -1, 0)$ and $(0, 1, 0)$ is almost spherically convex but obviously not strictly convex.

However, for $d = 2$, almost spherical convexity obviously implies strict convexity.

An example of a strictly convex but not almost spherically convex figure in R^2 is given in 3.7. (A similar example can be found in [8].)

A simple consequence of Theorem 1.3 and Theorem 1.6 below is also the following partial solution of the original Heil problem, see §4.

1.4. THEOREM. *Let $K \subset R^d$ be a compact d-dimensional reduced body. Suppose K is strictly convex and almost every point of ∂K has a neighbourhood within which ∂K is a C^2-hypersurface. (That takes place, say, when ∂K is piecewise C^2.) Then K is of constant width.*

1.5. It follows easily from [1, §2, §3] that almost every boundary point p of a compact d-dimensional convex body $C \subset R^d$ is regular and one can introduce in R^d Cartesian coordinates $\bar{x}$, z where $\bar{x} = (x_1, x_2, \ldots, x_{d-1})$ with the following properties.

(i) The point p is the origin of the system; the $\bar{x}$-plane coincides with the tangent plane at p and the z-axis shows towards int C.

(ii) A neighbourhood of p on the surface ∂C is given by the equation

$$z = z(\bar{x}) = \frac{1}{2} \sum_{i=1}^{d-1} k_i x_i^2 + \varepsilon \cdot |\bar{x}|^2 \tag{1.5.1}$$

where $\bar{x}$ varies in a domain D containing the origin, $z(\bar{x})$ is a convex function, $k_i \geqq 0$ and $\varepsilon = \varepsilon(\bar{x}) \xrightarrow[|\bar{x}| \to 0]{} 0$.

(iii) Let $z_i = z_i(\bar{x})$, $\bar{x} \in D$, $i = 1, 2, \ldots, d-1$, denote $\partial z / \partial x_i$ where it exists, and any value between the left and the right derivative $\partial z / \partial x_i$ inclusively where the derivative $\partial z / \partial x_i$ does not exist. Then $z_i(\bar{x})$ is differentiable at the origin $\bar{0}$ and

$$z_i(\bar{0}) = 0, \qquad \frac{\partial z_i}{\partial x_i}(\bar{0}) = k_i, \qquad \frac{\partial z_i}{\partial x_j}(\bar{0}) = 0 \tag{1.5.2}$$
$$\text{for } i = 1, 2, \ldots, d-1 \quad \text{and} \quad j \neq i.$$

Such a point p will be called a *C^2-point.*

1.6. Theorem. *Let $K \subset R^d$ be a compact d-dimensional reduced body of width W. Suppose there exists a dense set M in ∂K of regular points such that $a \in M$ implies that a is the only common point of K and the tangent plane at a. If K is supported by a sphere at a C^2-point $p \in \partial K$ then K has an R-support at p for any $R > W$.*

Theorem 1.6 is proved in 2.3 and 2.4

1.7. Theorem. *Let $K \subset R^d$ be a compact d-dimensional reduced body of width W. Suppose K has an R-support for any $R > W$ almost everywhere in ∂K. Then K is of constant width.*

Theorem 1.7 is proved in 3.6.

1.8. Proof of Theorem 1.3. According to 1.5, the set of C^2-points is of full measure in ∂A. The set of points where A is supported by a sphere is also of full measure. Therefore the intersection I of the two sets is of full measure. Clearly I can be regarded as the set M in Theorem 1.6. By Theorem 1.6, the body A has an R-support for any $R > W$ at any point $p \in I$. By Theorem 1.7, the body A is of constant width.

In 3.3, 3.4, we prove also the following auxiliary result.

1.9. THEOREM. *Let $C \subset R^d$ be a compact convex d-dimensional body and R be a positive number. Suppose C has an R-support almost everywhere in ∂C. Let p be an arbitrary point in ∂C and let H be a supporting hyperplane of C at p. Denote by B the compact ball of radius R tangent to H at p and lying with C on the same side of H. Then $C \subset B$.*

§2. Curvature of the boundary of a reduced body

2.1. We denote by xy both the closed segment with the ends x, y and its length. For a compact convex d-dimensional body $C \subset R^d$ and a unit vector u, we will denote by $w(C, u)$ the width of C in the direction u. Put $\Delta(C) = \min_{u \in S^{d-1}} w(C, u)$.

A segment pq will be called a *chord* of C if $p \in \partial C$, $q \in \partial C$. We will say that a chord pq is generated by a direction u if $p \in H_1$, $q \in H_2$ where H_1 and H_2 are the supporting hyperplanes orthogonal to u.

The following two remarks are well known.

REMARK A. Let C_x, $x \in [x_1, x_2]$, be a continuous family of compact convex d-dimensional sets in R^d and let $u \in S^{d-1}$ be a direction. Then $w(C_x, u)$ is a continuous function in the domain $[x_1, x_2] \times S^{d-1}$.

REMARK B. If $w(C, u) = \Delta(C)$ then among the chords generated by u there is one having direction u and length $\Delta(C)$. Such a chord is obviously unique if one of the two supporting planes orthogonal to u has only one common point with C. That point is an end of the chord.

2.2. LEMMA. *Let $K \subset R^d$ be a compact d-dimensional reduced body of width W. Let $p \in \partial K$ be a regular point such that the tangent plane T at p satisfies $T \cap K = p$. Denote by u the unit interior normal to ∂K at p. Then $w(K, u) = W$.*

PROOF. Suppose to the contrary that

$$w(K, u) > W. \tag{2.2.1}$$

Let T_x, $0 \leqq x \leqq W$, be the plane parallel to T, intersecting K and distant x from T. Let C_x be the compact part of K cut by T_x and satisfying int $C_x \not\ni p$.

Since K is reduced, there is a direction u_x such that

$$w(C_x, u_x) = \Delta(C_x) < W. \tag{2.2.2}$$

By Remark B in 2.1, C_x has a chord p_xq_x generated by u_x having direction u_x and length $\Delta(C_x) < W$. One of its ends, say p_x, belongs to T_x because otherwise $\Delta(C_x) = p_xq_x = w(K, u_x) \geqq W$ in contradiction to (2.2.2). One may assume that u_x shows from p_x to q_x.

Consider now a sequence $x_i \xrightarrow[i\to\infty]{} 0$, $0 < x_i < W$. The condition $T \cap K = p$ implies that $p_{x_i} \xrightarrow[i\to\infty]{} p$. Uniqueness of T implies easily that the supporting plane H_i at p_{x_i} orthogonal to u_{x_i} converges to T as $i \to \infty$. Therefore $u_{x_i} \xrightarrow[i\to\infty]{} u$. Since C_x, $x \in [0, W]$, is a continuous family and by Remark A in 2.1, one has

$$\lim_{i\to\infty} w(C_{x_i}, u_{x_i}) = w(K, u) > W$$

(under our assumption (2.2.1)). On the other hand, the limit cannot be greater than W due to (2.2.2). Thus Lemma 2.2 has been proved.

2.3. Proof of Theorem 1.6. Consider the representation (1.5.1) for ∂K in a neighbourhood of the point p. Obviously it is enough to show that $k_i \geqq 1/W$, $i = 1, 2, \ldots, d-1$. Suppose to the contrary that $k_1 < 1/W$. Take a point $a \in \partial K$ with coordinates $(t, 0, 0, \ldots, 0, z)$ where $t > 0$ and $z = z(t, 0, \ldots, 0)$. Select a sequence $a_i \xrightarrow[i\to\infty]{} a$, $a_i \in M$, such that the unit interior normal u_i to ∂K at the point a_i converges to a unit vector u as $i \to \infty$. The tangent plane T_i at a_i converges to a supporting plane H at the point a orthogonal to the vector u. Let

$$z = z(t, 0, \ldots, 0) + m_1(x_1 - t) + m_2x_2 + \cdots + m_{d-1}x_{d-1}$$

be an equation of H. Obviously $m_i = m_i(t)$ belongs to the closed interval between the left and the right derivative $\partial z(t, 0, \ldots, 0)/\partial x_i$. Therefore we can put $z_i(t, 0, \ldots, 0) = m_i$, $i = 1, 2, \ldots, d-1$, see 1.5(iii). At other points $\bar{x} \in D$, outside the positive part of the x_1-axis, we define $z_i(\bar{x})$ arbitrarily, in compliance with 1.5(iii).

By Lemma 2.2, $w(K, u_i) = W = \Delta(K)$. Since $K \cap T_i = a_i$ and by Remark B in 2.1, the body K has a chord a_ib_i of length W whose direction (from a_i to b_i) is u_i. Clearly a_ib_i converges to a chord ab of length W whose direction (from a to b) is u.

2.4. Denote by $h(t)$ the coordinate z of the point $b \in \partial K$. Obviously

$$\begin{aligned} h(t) &= z(t, 0, \ldots, 0) + W\Big/\sqrt{1 + \sum_{i=1}^{d-1} m_i^2} \\ &\geqq z(t, 0, \ldots, 0) + W - \frac{W}{2}\sum_{i=1}^{d-1} m_i^2. \end{aligned} \tag{2.4.1}$$

By (1.5.1),

$$z(t,0,\dots,0)=\tfrac{1}{2}k_1t^2+\varepsilon\cdot t^2 \qquad \text{where } \varepsilon=\varepsilon(t)\xrightarrow[t\to 0]{} 0. \tag{2.4.2}$$

Due to 1.5(iii),

$$m_i=z_i(t,0,\dots,0)=z_i(\bar{0})+\frac{\partial z_i}{\partial x_1}(\bar{0})t+\varepsilon_i t \qquad \text{where } \varepsilon_i=\varepsilon_i(t)\xrightarrow[t\to 0]{} 0;$$

$$|m_1-k_1t|<\tilde{\varepsilon}t, |m_2|<\tilde{\varepsilon}t,\dots,|m_{d-1}|<\tilde{\varepsilon}t \qquad \text{where } \tilde{\varepsilon}=\max_i|\varepsilon_i|. \tag{2.4.3}$$

Inserting (2.4.2) and (2.4.3) into (2.4.1), one has

$$\begin{aligned} h(t)&>\tfrac{1}{2}k_1t^2+\varepsilon t^2+W-\frac{W}{2}[k_1^2+2k_1\tilde{\varepsilon}+\tilde{\varepsilon}^2+(d-2)\tilde{\varepsilon}^2]t^2 \\ &=W+\frac{t^2}{2}[k_1(1-Wk_1)+2\varepsilon-2Wk_1\tilde{\varepsilon}-W(d-1)\tilde{\varepsilon}^2]. \end{aligned}$$

Since K is supported at p by a sphere, one has $k_1>0$. Due to our contrary assumption, $1-Wk_1>0$. Therefore $h(t)>W$ when $t>0$ is sufficiently small. On the other hand, $h(t)\leqq W$ since the width of K in the direction of the z-axis is W according to Lemma 2.2.

§3. Global R-support

3.1. Lemma. *Let $y(x)$ be a convex function in a segment $|x-t|\leqq\delta$ and let $y''(t)$ exist. Suppose the convex hull of the graph of $y(x)$ has an R-support in the xy-plane at the point $(t,y(t))$. Then*

$$y''(t)\geqq\frac{1}{R}(1+y'^2(t))^{3/2}. \tag{3.1.1}$$

3.2. Proof. Having an R-support means obviously that for any $\Delta>0$,

$$y(x)>y(t)+y'(t)(x-t)+\left[\frac{1}{R}(1+y'^2(t))^{3/2}-\Delta\right]\frac{(x-t)^2}{2} \tag{3.2.1}$$

when $|x-t|\neq 0$ is sufficiently small. Since $y(x)$ is convex, it can be represented in the form $y(x)=y(t)+\int_t^x\phi(z)dz$, see [6, p. 304]. Therefore, and by [7, p. 337, Theorem 2], $y(x)$ is absolutely continuous (and thus y' exists almost everywhere). Now by Lebesgue's Theorem [7, p. 338],

$$y(x)=y(t)+\int_t^x y'(z)dz. \tag{3.2.2}$$

For those x's where y' exists,

$$y'(x) = y'(t) + y''(t)(x-t) + \varepsilon \cdot (x-t) \qquad \text{where } \varepsilon = \varepsilon(x) \xrightarrow[x \to t]{} 0. \tag{3.2.3}$$

Suppose now that (3.1.1) is wrong. Then

$$y''(t) < \frac{1}{R}(1 + y'^2(t))^{3/2} - 2\Delta \qquad \text{for some } \Delta > 0.$$

For $x \neq t$ sufficiently close to t, one has $|\varepsilon(x)| < \Delta$ and by (3.2.3),

$$y'(x) < y'(t) + \left[\frac{1}{R}(1 + y'^2(t))^{3/2} - 2\Delta\right](x-t) + \Delta \cdot (x-t).$$

Along with (3.2.2), that implies

$$y(x) < y(t) + y'(t)(x-t) + \left[\frac{1}{R}(1 + y'^2(t))^{3/2} - \Delta\right]\frac{(x-t)^2}{2},$$

which contradicts (3.2.1).

3.3. Proof of Theorem 1.9. Consider first the case $d = 2$. Suppose to the contrary that $C \not\subset B$. Clearly, there exists a compact circle $B' \supset B$ of radius $R' > R$ tangent to H at p and such that $C \not\subset B'$.

Select a point $q \in \partial C$ as follows. If there exists an arc $\breve{pm} \subset \partial C$, $m \neq p$, satisfying $\breve{pm} \cap \operatorname{int} B' = \varnothing$ then put $q = p$. Otherwise one can easily find an arc $\breve{ab} \subset \partial C$ such that $a \in \partial B'$, $b \in \partial B'$, $(\breve{ab} \setminus \{a, b\}) \subset \operatorname{int} B'$ and $\breve{ab}$ belongs to that segment of the circle B' bounded by the chord ab whose central angle is $\leqq \pi$. As one moves the circle B' in the direction orthogonal to the chord ab and showing from ab to the center of B', there will be a "moment when $\partial B'$ intersects the arc $\breve{ab}$ for the last time." Let $q \in \breve{ab}$ be a point of that intersection.

It is easy to see that, in both cases, the selected point $q \in \partial C$ satisfies the following conditions.

(A) There exists an arc $\breve{qm} \subset \partial C$ such that $\breve{qm} \cap \operatorname{int} B' = \varnothing$.

(B) The direction of the arc $\breve{qm}$ at q is tangent to $\partial B'$.

3.4. Introduce now Cartesian coordinates in R^2 with the origin at q, the x-axis showing in the direction of the arc $\breve{qm}$ and the y-axis towards the center of B'. Due to the condition (B) in 3.3, a part of the arc $\breve{qm}$ can be given by an equation $y = y(x)$, $x \in [0, x_0]$, where $y(x)$ is a convex function with $y(0) = y'(0) = 0$. Condition 3.3(A) implies that for any $\Delta > 0$,

$$(3.4.1) \qquad y(x) < \left(\frac{1}{R'} + \Delta\right)\frac{x^2}{2}$$

when $x > 0$ is sufficiently small.

As in 3.2, one can see that y' exists almost everywhere and

$$(3.4.2) \qquad y(x) = \int_0^x y'(t)dt, \qquad x \in [0, x_0].$$

Since $y(x)$ is convex, $y'(x)$ is non-decreasing and thus y'' exists almost everywhere, see [7, Theorem 1, p. 320]. By [7, Theorem 1, p. 333],

$$(3.4.3) \qquad y'(x) \geqq \int_0^x y''(t)dt.$$

Recall that C has an R-support almost everywhere in ∂C. Therefore, at almost every point t in the segment $[0, x_0]$, both conditions hold simultaneously: $y''(t)$ exists and C has an R-support at its boundary point $(t, y(t))$. Lemma 3.1 and (3.4.3) imply now

$$y'(x) \geqq \int_0^x \frac{1}{R}(1 + y'^2(t))^{3/2} dt \geqq \int_0^x \frac{dt}{R} = \frac{x}{R}.$$

By (3.4.2),

$$y(x) \geqq \int_0^x \frac{t}{R}\, dt = \frac{x^2}{2R}.$$

That contradicts (3.4.1) when $\Delta < 1/R - 1/R'$.

3.5. We prove now Theorem 1.9 for $d > 2$. Suppose to the contrary that there exists a point $q \in C \setminus B$. Denote by S^{d-2} the set of directions (unit vectors) parallel to the supporting hyperplane H. Let P_u, $u \in S^{d-2}$, be the 2-plane containing the point p, the center of the ball B and parallel to u. Let $v \in S^{d-2}$ be such that $P_v \ni q$. The fact that C has R-support almost everywhere in ∂C and Fubini's theorem imply easily that in any neighbourhood of v there exists a direction $u \in S^{d-2}$ such that C has R-support almost everywhere in the curve $\partial C \cap P_u$.

Note that if C is supported at a point $x \in \partial C \cap P_u$ by a sphere Q of radius R then the plane figure $C \cap P_u$ is supported at x by the circumference $Q \cap P_u$ of a radius $R' \leqq R$ (a version of Meusnier's Theorem). Therefore $C \cap P_u$ has R'-support and consequently R-support at x. Thus the compact convex plane figure $C \cap P_u$ has R-support almost everywhere in its boundary. By Theorem

1.9 for $d = 2$, the figure belongs to the circle $B \cap P_u$. On the other hand, this is impossible when u is close enough to v since $C \cap P_v \not\subset B \cap P_v$.

3.6. Proof of Theorem 1.7. Suppose to the contrary that diameter D of K satisfies $D > W$. Fix a chord ab such that $ab = D$. Since K is supported by a sphere almost everywhere in ∂K and almost every point in ∂K is regular (see 1.5), there exists a regular point $q \in \partial K$ with a spherical support satisfying $qb < (D - W)/3$. By Lemma 2.2 and Remark B in 2.1, there exists the (unique) point $p \in \partial K$ such that $pq = W$ and the hyperplane H through p orthogonal to pq is a supporting hyperplane of K.

Put $R = W + (D - W)/3$ and let $z \in R^d$ satisfy $pz = R$, $pz \supset pq$. Since K has R-support almost everywhere and by Theorem 1.9, K belongs to the ball of radius R centered at z (and thus tangent to H). Therefore $za \leqq R$. One has now

$$D = ab \leqq az + zq + qb < R + \frac{D - W}{3} + \frac{D - W}{3} = D.$$

This contradiction completes the proof.

3.7. We produce now the example of a strictly convex but not almost spherically convex figure $F \subset R^2$ mentioned in 1.3. Take a set $S \subset [0,1]$ of a positive measure m such that for any segment $[a, b] \subset [0, 1]$, the measure of $S \cap [a, b]$ is less than $b - a$. (S could be a Cantor set under a proper selection of the length of the "thrown away" intervals.) Define f: $[0,1] \rightarrow R$ as follows. Put $f(x) = 0$ for $x \in S$ and $f(x) = 1$ for $x \in [0,1] \setminus S$. Let

$$\phi(x) = \int_0^x f(z)dz, \qquad y(x) = \int_0^x \phi(z)dz.$$

The function $y(x)$ is convex since $\phi(x)$ is nondecreasing. By [7, p. 337, Theorem 2], $\phi(x)$ is absolutely continuous (and thus ϕ' exists almost everywhere). By the Lebesgue Theorem [7, p. 338], $\phi(x) = \int_0^x \phi'(z)dz$. Along with the definition of ϕ, this implies that $\phi'(x) = f(x)$ almost everywhere in $[0, 1]$, see [7, Theorem 1, p. 330]. Continuity of ϕ means also that $y'(x) = \phi(x)$ for any $x \in [0, 1]$. Thus $y''(x) = f(x)$ almost everywhere in $[0, 1]$. Therefore $y''(x) = 0$ almost everywhere in S.

Since $m > 0$, the set $S^* \overset{\text{def}}{=} \{(x, y(x))/y''(x) = 0\}$ has a positive measure on the graph of $y(x)$. Let now F be a convex figure bounded by that graph and an arc of a circumference. Due to (3.1.1), F has no R-support at any point of S^*. Thus, F is not almost spherically convex.

Clearly, ∂F contains no segments: otherwise, almost everywhere in a segment

$[a, b] \subset [0,1]$, one would have $f(x) = y''(x) = 0$ which contradicts the definition of f. Thus F is strictly convex.

§4. The case of a good boundary

4.1. PROOF OF THEOREM 1.4. Denote by F the set of points having a neighbourhood in which ∂K is C^2. Let $k_1, k_2, \ldots, k_{d-1}$ be the principal curvatures in F. We show first that the set of the points from F, where $k_1 k_2 \cdots k_{d-1} > 0$, is dense in F. Suppose to the contrary that $k_1 k_2 \cdots k_{d-1} = 0$ in a closed neighbourhood $N \subset F$ of a point $m \in F$. Due to strict convexity of K, the spherical representation $s\colon F \to S^{d-1}$ is one-to-one. Therefore the distance ε (measured in S^{d-1}) between $s(m)$ and $s(\partial N)$ is positive. Clearly $s(N)$ contains the metric ball of radius ε centered at $s(m)$ and thus has a positive $(d-1)$-area A. On the other hand,

$$A = \int_N k_1 k_2 \cdots k_{d-1} dA = 0.$$

4.2. Obviously K is supported by a sphere at each point of F with $k_1 k_2 \cdots k_{d-1} > 0$. By Theorem 1.6, K has an R-support at such a point for any R greater than width W of K. Therefore $k_i \geqq 1/W$, $i = 1, 2, \ldots, d-1$, at this point. By continuity of Gauss curvature and due to 4.1, the curvature $k_1 k_2 \cdots k_{d-1} \geqq W^{-d+1}$ everywhere in F. Thus K is supported by a sphere everywhere in F. By Theorem 1.3, K is of constant width.

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